

# The Riesz hull of a semisimple MV-algebra

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*Dedicated to Prof. Antonio Di Nola on the occasion of his 65th birthday.*

## Abstract

MV-algebras and Riesz MV-algebras are categorically equivalent to abelian lattice-ordered groups with strong unit and, respectively, with Riesz spaces (vector-lattices) with strong unit. A standard construction in the literature of lattice-ordered groups is the *vector-lattice hull* of an archimedean lattice-ordered group. Following a similar approach, in this paper we define the *Riesz hull* of a semisimple MV-algebra.

## 1 Introduction

MV-algebras were first defined by Chang [4] as algebraic structures corresponding to the  $\infty$ -valued Łukasiewicz logic. An *MV-algebra* is a structure  $(A, \oplus, *, 0)$ , where  $(A, \oplus, 0)$  is an abelian monoid and the following identities hold for all  $x, y \in A$ :  $(x^*)^* = x$ ,  $0^* \oplus x = 0^*$  and  $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$ . One of the main engines of MV-algebra theory is the categorical equivalence between MV-algebras and abelian lattice-ordered groups with strong unit [18]. As a consequence, any MV-algebra is isomorphic to the unit interval  $[0, u]$  of an abelian lattice-ordered group  $(G, u)$ , with operations defined by  $x^* = u - x$  and  $x \oplus y = u \wedge (x + y)$ . MV-algebras stand to Łukasiewicz logic as boolean algebras stand to classical logic: an equation holds in any MV-algebra if and only if it holds in the real interval  $[0, 1]$  endowed with the following operations

$$x \oplus y = \min\{1, x + y\} \text{ and } x^* = 1 - x,$$

for every  $x, y \in [0, 1]$ . The real interval  $[0, 1]$  with the above operations is the *standard MV-algebra* and it is usually denoted by  $[0, 1]_{MV}$ .

Adding a product operation to the signature of MV-algebras was a natural step, which led to fruitful results, both in logic and algebra. Once the MV-algebra structure is enriched, categorical equivalences with particular lattice-ordered structures are proved.

*PMV-algebras* are defined in [10] as MV-algebras endowed with a product operation  $\cdot : A \times A \rightarrow A$ , satisfying some particular identities. The category of PMV-algebras is equivalent with the category of lattice-ordered rings with strong unit. In [11] the internal product is replaced by a scalar multiplication with scalars from  $[0, 1]$ , so MV-algebras are endowed with a map  $\cdot : [0, 1] \times A \rightarrow A$ . The structures obtained in this way are called *Riesz MV-algebras* and they are categorically equivalent with Riesz spaces (vector-lattices) with strong unit. The real interval  $[0, 1]$  endowed with the natural product generates the variety of Riesz MV-algebras. Note that, in the case of PMV-algebras,  $[0, 1]$  generates only a proper quasi-variety [17].

A standard construction in the literature of lattice-ordered groups is the *vector-lattice hull* of an archimedean lattice-ordered group, defined by Conrad in [7] and further analyzed by Bleier in [3]. We also refer to [16] for an extensive treatment of hull classes for archimedean lattice-ordered groups.

We briefly remind Conrad's definition. If  $G$  is an archimedean lattice-ordered group, then the *v-hull* of  $G$  is a vector-lattice  $U$  such that  $G$  is an essential subgroup of  $U$  and no proper  $\ell$ -subspace of  $U$  contains  $G$ . Assume  $G_d$  is the divisible hull of  $G$  and  $\hat{G}_d$  is the Dedekind-MacNeille completion of  $G_d$ . Hence the vector-lattice generated by  $G_d$  in  $\hat{G}_d$ , denoted by  $\mathbf{R}(G)$ , is the *v-hull* of  $G$ . Moreover, Bleier proved that the correspondence  $G \mapsto \mathbf{R}(G)$  is functorial.

In this paper we investigate a similar construction for semisimple MV-algebras and semisimple Riesz MV-algebras. If  $A$  is a semisimple MV-algebra we say that a Riesz MV-algebra  $U$  is the *Riesz hull* of  $A$  if  $A$  is essentially embedded in  $U$  and  $A$  is a set of generators for  $U$ . In Section 4 we prove that *any semisimple MV-algebra has a Riesz hull*. Moreover, the Riesz hull of the free MV-algebra over a set  $X$  is the free Riesz MV-algebra over  $X$ . In Section 5 we prove that the construction of the Riesz hull is functorial. Moreover, the hull functor commutes with the categorical equivalences between the corresponding classes of MV-algebras and lattice-ordered groups.

We chose to make direct proofs in the theory of MV-algebras. Alternative proofs can be given using Conrad's construction and various preservation properties of the categorical equivalence between MV-algebras and lattice-ordered groups, but we find the direct approach more relevant for our purpose.

In Section 2 and 3 we recall the basic results on MV-algebras and Riesz MV-algebras that are required for our development. We refer to [8] for background knowledge on lattice-ordered groups, to [15] for Riesz spaces and to [6] for universal algebra.

## 2 MV-algebras

**Definition 2.1.** An *MV-algebra* is a structure  $(A, \oplus, *, 0)$  of type  $(2,1,0)$  which satisfies the following:

(MV1)  $(A, \oplus, 0)$  is an abelian monoid,

(MV2)  $(a^*)^* = a$ ,

(MV3)  $0^* \oplus a = 0^*$ ,

(MV4)  $(a^* \oplus b)^* \oplus b = (b^* \oplus a)^* \oplus a$ ,

for any  $a, b \in A$ .

We refer to [5] for all the unexplained notions related to MV-algebras.

In any MV-algebra  $A$  we can define the following:

$$\begin{aligned} 1 &\stackrel{def}{=} 0^*, & a \odot b &\stackrel{def}{=} (a^* \oplus b^*)^*, \\ a \vee b &\stackrel{def}{=} (a \odot b^*) \oplus b, & a \wedge b &\stackrel{def}{=} (a \oplus b^*) \odot b, \end{aligned}$$

for any  $a, b \in A$ . Hence  $(A, \vee, \wedge, 0, 1)$  is a bounded distributive lattice such that

$$a \leq b \text{ if and only if } a \odot b^* = 0.$$

The notions of MV-homomorphism and MV-subalgebra are defined as usual.

We recall that a *lattice-ordered group* (an  $\ell$ -group) is a structure  $(G, +, 0, \leq)$  such that  $(G, +, 0)$  is a group,  $(G, \leq)$  is a lattice and any group translation is isotone [8]. An element  $u \in G$  is a *strong unit* if  $u \geq 0$  and for any  $x \in G$  there is a natural number  $n$  such that  $x \leq nu$ . An  $\ell u$ -group will be an abelian  $\ell$ -group which has a strong unit. If  $(G, u)$  is an  $\ell u$ -group, we define

$$\begin{aligned} [0, u] &= \{x \in G \mid 0 \leq x \leq u\} \text{ and} \\ x \oplus y &= (x + y) \wedge u, \quad x^* = u - x, \text{ for any } x, y \in [0, u]. \end{aligned}$$

Then  $[0, u]_G = ([0, u], \oplus, *, 0)$  is an MV-algebra.

We denote by  $\mathcal{MV}$  the category of MV-algebras and by  $\mathcal{AG}_u$  the category of unital abelian lattice-ordered groups with unit-preserving  $\ell$ -morphisms. In [18] the functor  $\Gamma: \mathcal{AG}_u \rightarrow \mathcal{MV}$  is defined as follows:

$$\begin{aligned} \Gamma(G, u) &= [0, u]_G, \text{ for any unital } \ell\text{-group } (G, u), \\ \Gamma(f) &= f|_{[0, u]}, \text{ for any } \ell\text{-morphism } f: (G, u) \rightarrow (G', u') \text{ from } \mathcal{AG}_u. \end{aligned}$$

**Theorem 2.1.** [18] *The functor  $\Gamma$  establishes a categorical equivalence between  $\mathcal{AG}_u$  and  $\mathcal{MV}$ .*

The standard MV-algebra is  $[0, 1] = \Gamma(\mathbb{R}, 1)$ .

**Theorem 2.2.** [4] *An equation holds in  $[0, 1]$  if and only if it holds in any MV-algebra.*

As a consequence, the variety of MV-algebras is generated by  $[0, 1]$ .

**Theorem 2.3.** [9] *Any MV-algebra  $A$  is isomorphic with an algebra of  $^*[0, 1]$ -valued functions, where  $^*[0, 1]$  is the unit interval of the lattice-ordered group of nonstandard reals  $^*\mathbb{R}$ .*

If  $A$  is an MV-algebra,  $a \in A$  and  $n \geq 0$  is a natural number, we define

$$0a \stackrel{\text{def}}{=} 0 \text{ and } na \stackrel{\text{def}}{=} (n-1)a \oplus a, \text{ if } n > 0.$$

**Definition 2.2.** If  $\iota : A \rightarrow B$  is an MV-embedding then we say that:

- (1)  $\iota$  is *order dense* if for any  $b > 0$  in  $B$ , there exists  $a > 0$  in  $A$  such that  $\iota(a) \leq b$ ,
- (2)  $\iota$  is *essential* if for any  $b > 0$  in  $B$ , there exists  $a > 0$  in  $A$  such that  $\iota(a) \leq nb$ , for some natural number  $n \geq 0$ .

For any MV-algebra  $A$ , a nonempty set  $I \subseteq A$  is an *MV-ideal* if the following hold:

- (I1)  $a \leq b$  and  $b \in I$  implies  $a \in I$ ,
- (I2)  $a, b \in I$  implies  $a \oplus b \in I$ .

**Remark 2.1.** An embedding  $\iota : A \rightarrow B$  is essential if and only if for any ideal  $I$  of  $B$ ,  $I \neq \{0\}$  implies  $I \cap \iota(A) \neq \{0\}$ .

**Lemma 2.1.** *Let  $\iota : A \rightarrow B$  be an essential embedding. If  $C$  is an MV-algebra and  $f_A : A \rightarrow C$ ,  $f_B : B \rightarrow C$  are MV-homomorphisms such that  $f_B \circ \iota = f_A$  and  $f_A$  is an embedding then  $f_B$  is an embedding.*

*Proof.* Assume  $b \in B$  such that  $f_B(b) = 0$ . If  $b \neq 0$  there is  $a > 0$  in  $A$  such that  $a \leq nb$ , so  $f_A(a) = f_B(\iota(a)) = 0$ . Since  $f_A$  is an embedding we infer that  $a = 0$ , which is a contradiction, so  $b = 0$  and  $f_B$  is an embedding.  $\square$

An ideal  $I$  of  $A$  is *proper* if  $I \neq A$ . A *maximal ideal* is a maximal element of the set of proper ideals ordered by inclusion. We denote by  $\text{Max}(A)$  the set of all maximal ideals of  $A$ . Remember that, for any MV-algebra  $A$ ,  $\text{Max}(A)$  endowed with the spectral topology is a compact and Hausdorff space [5].

An MV-algebra is *semisimple* if  $\bigcap \{I \mid I \in \text{Max}(A)\} = \{0\}$ .

Recall that an  $\ell u$ -group  $(G, u)$  is *archimedean* if, for any  $x, y \in G$ , we have

$$nx \leq y, \text{ for any } n \in \mathbb{N}, \text{ implies } x \leq 0.$$

**Remark 2.2.** [5] Let  $A$  be an MV-algebra and  $(G, u)$  an  $\ell u$ -group such that  $A \simeq \Gamma(G, u)$ . Then  $A$  is semisimple if and only if  $G$  is archimedean.

The semisimple MV-algebras are the algebras of  $[0, 1]$ -valued functions, i.e. for any semisimple MV-algebra  $A$  there exists a set  $X$  such that  $A$  is isomorphic with a subalgebra of  $[0, 1]^X$  [2]. If  $X$  is a topological space, we set  $C(X) = \{f : X \rightarrow [0, 1] \mid f \text{ continuous}\}$ , which obviously is a semisimple MV-algebra.

**Theorem 2.4.** [5] Any semisimple MV-algebra  $A$  is isomorphic with a separating subalgebra of  $C(\text{Max}(A))$ .

For a semisimple MV-algebra  $A$  we denote by  $\mathbf{A}$  the subalgebra of  $C(\text{Max}(A))$  such that  $A \simeq \mathbf{A}$  and by  $\varphi_A : A \rightarrow \mathbf{A}$  the corresponding isomorphism.

**Definition 2.3.** An MV-algebra  $A$  is *divisible* if for any element  $a \in A$  and  $n > 1$  in  $\mathbb{N}$  there exists  $x \in A$  such that  $nx = a$  and  $(n-1)x \leq x^*$ .

We refer to [14] for a systematic investigation of the divisible MV-algebras and their logic.

**Remark 2.3.** [14] An  $\ell$ -group  $G$  is *divisible* if for any element  $g \in G$  and any  $n > 1$  in  $\mathbb{N}$  there exists  $x \in G$  such that  $nx = g$ . One can easily see that an  $\ell u$ -group  $(G, u)$  is divisible if and only if the MV-algebra  $[0, u]_G$  is divisible.

If  $X$  is a compact Hausdorff space then  $C(X, \mathbb{R}) = \{f : X \rightarrow \mathbb{R} \mid f \text{ continuous}\}$  is an  $\ell$ -group and the constant function  $\mathbf{1}$  is a strong unit.

**Remark 2.4.** It is well-known that any MV-algebra can be embedded in a divisible one (see, for example, [12]). We provide the details of this embedding for the semisimple case, which is relevant for our paper.

Assume  $(G, \mathbf{1})$  is an  $\ell u$ -subgroup of  $(C(X, \mathbb{R}), \mathbf{1})$  and  $A = [0, \mathbf{1}]_G \subseteq C(X)$ . We define  $G_d = \{\frac{g}{n} \mid g \in G, n \in \mathbb{N}, n \neq 0\}$  and  $A_d = [0, \mathbf{1}]_{G_d}$ . Hence  $G_d$  is a divisible  $\ell$ -group and  $A_d$  is a divisible MV-algebra. Let  $g \in G$  and  $n \in \mathbb{N}$  such that  $\frac{g}{n} \in A_d$ . It follows that  $0 \leq g \leq n\mathbf{1}$  in  $G$ , so there are  $a_1, \dots, a_n \in A$  such that  $g = a_1 + \dots + a_n$ . Hence  $\frac{g}{n} = \frac{a_1}{n} + \dots + \frac{a_n}{n}$ . In consequence

$$A_d = \{a \in C(X) \mid a = \frac{a_1}{n} + \dots + \frac{a_n}{n} \text{ for some } n \in \mathbb{N}, n \neq 0 \text{ and } a_1, \dots, a_n \in A\},$$

and it is straightforward that  $A \subseteq A_d$ .

If  $X$  is a compact Hausdorff space and  $\mathbf{A} \leq C(X)$  is a semisimple MV-algebra then we get an embedding  $\iota_{A,d} : \mathbf{A} \rightarrow \mathbf{A}_d$ .

We note that

**Lemma 2.2.** Under the above hypothesis, the following properties hold.

(a) The embedding  $\iota_{A,d}$  is essential.

(b) If  $U$  is a semisimple divisible MV-algebra and  $f : \mathbf{A} \rightarrow U$  is an MV-homomorphism then there exists a unique MV-homomorphism  $f_d : \mathbf{A}_d \rightarrow U$  such that  $f_d(a) \circ \iota_{A,d} = f$ . Moreover, if  $f$  is an embedding then  $f_d$  is also an embedding.

*Proof.* (a) follows easily from the description of  $\mathbf{A}_d$  from Remark 2.4.

(b) Assume that  $G$  and  $G_d$  are the  $\ell u$ -groups from Remark 2.4. One can easily see that whenever  $(H, v)$  is a divisible  $\ell u$ -group and  $h : G \rightarrow H$  is an  $\ell u$ -morphism there exists a unique  $\ell u$ -morphism  $h^\# : G_d \rightarrow H$  extending  $h$ , which is simply defined by  $h^\#(\frac{g}{n}) = \frac{h(g)}{n}$  for any  $g \in G$  and  $n \in \mathbb{N}$ . Hence the extension result for MV-algebras follows using the functor  $\Gamma$ . The embeddings are preserved by (a) and Lemma 2.1.  $\square$

Recall that an MV-algebra  $A$  is *complete* if any subset  $\{a_i \mid i \in I\}$  of  $A$  has infimum and supremum.

**Definition 2.4.** For a semisimple MV-algebra  $A$  we say that  $\hat{A}$  is the *Dedekind-MacNeille completion* of  $A$  if  $A \leq \hat{A}$ ,  $\hat{A}$  is complete and for any element  $\hat{a} \in \hat{A}$  there exists a family  $\{a_i \mid i \in I\} \subseteq A$  such that  $\hat{a} = \bigvee \{a_i \mid i \in I\}$ .

Since any complete MV-algebra is semisimple [5, Proposition 6.6.2], only semisimple MV-algebras admit completions.

**Remark 2.5.** Any semisimple MV-algebra  $A$  has a Dedekind-MacNeille completion  $\hat{A}$ , which is unique up to isomorphism. Moreover,  $A$  is order dense in  $\hat{A}$ .

We refer to [1] for a study of completions in the theory of MV-algebras, with a special focus on the Dedekind-MacNeille completion.

### 3 Riesz MV-algebras

Any MV-algebra is isomorphic with the unit interval of an  $\ell u$ -group. If we consider a Riesz space with strong unit instead of an  $\ell u$ -group, then the unit interval is closed under the scalar multiplication with scalars from  $[0, 1]$ . The structures obtained in this way are studied in [11].

**Definition 3.1.** [11] A *Riesz MV-algebra* is a structure  $(V, \cdot, \oplus, *, 0)$ , where  $(V, \oplus, *, 0)$  is an MV-algebra and  $\cdot : [0, 1] \times V \rightarrow V$  is a function such that:

$$(RMV1) \quad r \cdot (a \odot b^*) = (r \cdot a) \odot (r \cdot b)^*,$$

$$(RMV2) \quad (\max(r - q, 0)) \cdot a = (r \cdot a) \odot (q \cdot a)^*,$$

$$(RMV3) \quad (r \cdot q) \cdot a = r \cdot (q \cdot a),$$

$$(RMV4) \quad 1 \cdot a = a,$$

for any  $r, q \in [0, 1]$  and  $a, b \in V$ .

In order to simplify the notation, we shall frequently write  $ra$  instead of  $r \cdot a$ , for any  $r \in [0, 1]$  and  $a \in V$ . For a Riesz MV-algebra  $(V, \cdot, \oplus, *, 0)$  we denote by  $\mathbf{U}(V) = (V, \oplus, *, 0)$  its *MV-algebra reduct*.

**Remark 3.1.** [11] If  $V$  is a Riesz MV-algebra and  $I \subseteq \mathbf{U}(V)$  is an MV-algebra ideal, then  $r \cdot a \in I$  for any  $r \in [0, 1]$  and  $a \in I$ . Hence a Riesz MV-algebra has the same theory of ideals as its MV-algebra reduct. In consequence, a Riesz MV-algebra is semisimple if and only if its MV-algebra reduct is semisimple.

**Proposition 3.1.** [11] If  $V_1$  and  $V_2$  are Riesz MV-algebras and  $f : \mathbf{U}(V_1) \rightarrow \mathbf{U}(V_2)$  is an MV-homomorphism, then  $f(ra) = rf(a)$ , for any  $r \in [0, 1]$  and  $a \in V_1$ .

**Remark 3.2.** [11] By the previous proposition, it follows that *Riesz MV-algebra homomorphisms* are just MV-homomorphisms between Riesz MV-algebras, so we shall only state that a function is an MV-homomorphism, even if the domain and the codomain are Riesz MV-algebras.

We recall that a *Riesz space (vector-lattice)* [15] is a structure  $(L, \cdot, +, 0, \leq)$  such that  $(V, +, 0, \leq)$  is an abelian  $\ell$ -group,  $(V, \cdot, +, 0)$  is a real vector space and, in addition,  $x \leq y$  implies  $r \cdot x \leq r \cdot y$ , for any  $x, y \in L$  and  $r \in \mathbb{R}$ ,  $r \geq 0$ . A Riesz space is *unital* if the underlying  $\ell$ -group is unital. If  $(L, u)$  is a Riesz space with strong unit, then we denote by  $\Gamma_R(L, u) = ([0, u], \cdot, \oplus, *, 0)$ , where  $\cdot$  is the scalar multiplication restricted to scalars from  $[0, 1]$ .

**Remark 3.3.** [11] For any unital Riesz space  $(L, u)$ , the structure  $\Gamma_R(L, u)$  is a Riesz MV-algebra.

In this way we can define a functor  $\Gamma_R : \mathcal{RS}_u \rightarrow \mathcal{RMV}$ , where  $\mathcal{RS}_u$  is the category of unital Riesz spaces and  $\mathcal{RMV}$  is the category of Riesz MV-algebras. The categorial equivalence from Theorem 2.1 leads to the following one.

**Theorem 3.1.** [11] The functor  $\Gamma_R$  establishes a categorial equivalence.

The standard Riesz MV-algebra is  $([0, 1], \cdot, \oplus, *, 0)$  where  $([0, 1], \oplus, *, 0)$  is the standard MV-algebra and  $\cdot$  is the product of real numbers.

**Theorem 3.2.** [11] The variety of Riesz MV-algebras is generated by  $[0, 1]$ .

**Lemma 3.1.** The Dedekind-MacNeille completion of a semisimple divisible MV-algebra  $D$  is a Riesz MV-algebra  $\hat{D}$  in which  $D$  is order dense.

*Proof.* It is a straightforward consequence of the fact that the functor  $\Gamma$  preserves both divisibility [14] and completeness [13]. Hence there exists a divisible  $\ell u$ -group  $(G, u)$  such that  $D \simeq \Gamma(G, u)$  and  $\hat{D} = \Gamma(\hat{G}, u)$ , where  $\hat{G}$  is the Dedekind-MacNeille completion of  $G$ . Now we use the fact that the Dedekind-MacNeille completion of a divisible abelian  $\ell$ -group is a Riesz space [8]. The result can be directly proved by setting  $ra = \bigvee \{qa \mid q \in [0, 1] \cap \mathbb{Q}, q \leq r\}$  for any  $r \in [0, 1]$  and  $a \in \hat{D}$ .  $\square$

**Remark 3.4.** By Theorem 2.3, for any MV-algebra  $A$  there exists a set  $X$  such that  $A$  is embedded in the MV-algebra  $(*[0, 1])^X$ . Since  $*[0, 1]$  is obviously a Riesz MV-algebra, one can easily see that  $(*[0, 1])^X$  becomes a Riesz MV-algebra with the scalar multiplication defined componentwise. Hence any MV-algebra can be embedded in a Riesz MV-algebra.

In the following we prove that, for a semisimple MV-algebra  $A$ , we can define a unique (up to isomorphism) Riesz MV-algebra in which  $A$  is essentially embedded and we will further analyze the properties of this embedding.

## 4 The Riesz MV-algebra hull

In the sequel, we follow closely the similar construction for archimedean  $\ell$ -groups from [7] and [3], but our proofs are made directly in the context of MV-algebras.

Due to Remark 3.2, in the rest of this paper we will make no distinction between MV-homomorphisms and Riesz MV-algebra homomorphisms. If  $A$  is an MV-algebra and  $X$  is a subset of  $A$ , we shall denote by  $\langle X \rangle_{MV}$  the MV-subalgebra generated by  $X$  in  $A$ . Similarly, if  $V$  is a Riesz MV-algebra and  $X$  is a subset of  $V$ , we shall denote by  $\langle X \rangle_{RMV}$  the Riesz MV-subalgebra generated by  $X$  in  $V$ .

If  $A$  is a semisimple MV-algebra, then its divisible hull  $A_d$  is also semisimple. If  $X = \text{Max}(A_d)$  is the compact Hausdorff space of the maximal ideals of  $A_d$ , then

$$A \simeq \mathbf{A} \subseteq \mathbf{A}_d \subseteq C(X).$$

Let  $\hat{\mathbf{A}}_d$  be the Dedekind-MacNeille completion of  $\mathbf{A}_d$ . By Lemma 3.1,  $\hat{\mathbf{A}}_d$  is a Riesz MV-algebra. We denote by  $\mathbf{R}(A)$  the Riesz MV-algebra generated by  $\mathbf{A}$  in  $\hat{\mathbf{A}}_d$ .

For a semisimple MV-algebra  $A$ , we assume the following:

$\varphi_A : A \rightarrow \mathbf{A}$  is the canonical MV-algebra isomorphism,

$\iota_{A,d} : \mathbf{A} \rightarrow \mathbf{A}_d$  is the embedding of  $\mathbf{A}$  in its divisible hull  $\mathbf{A}_d$ ,

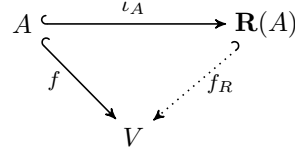
$\hat{\iota}_{A,d} : \mathbf{A}_d \rightarrow \hat{\mathbf{A}}_d$  is the embedding of  $\mathbf{A}_d$  in its Dedekind-MacNeille completion.



Hence we denote by  $\iota_A : A \rightarrow \mathbf{R}(A)$  the co-restriction to  $\mathbf{R}(A)$  of the homomorphism  $\hat{\iota}_{A,d} \circ \iota_{A,d} \circ \varphi_A$ .

**Theorem 4.1.** *If  $A$  is a semisimple MV-algebra and  $\mathbf{R}(A)$  is defined as above, then the following properties hold.*

- (a) *There exists an embedding  $\iota_A : A \rightarrow \mathbf{R}(A)$  and  $\mathbf{R}(A) = \langle \iota_A(A) \rangle_{RMV}$ .*
- (b) *The embedding  $\iota_A$  is essential.*
- (c) *If  $V$  is a semisimple Riesz MV-algebra and  $f : A \rightarrow V$  is an MV-embedding then there exists an MV-embedding  $f_R : \mathbf{R}(A) \rightarrow V$  such that  $f_R(\iota_A(a)) = f(a)$ , for any  $a \in A$ .*



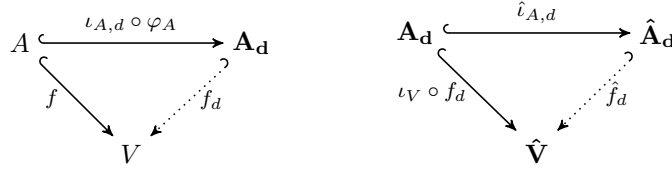
*Proof.*

- (a) follows by definition, since  $A$  is embedded in  $\mathbf{A}_d$  and  $\mathbf{A}_d$  is embedded in  $\hat{\mathbf{A}}_d$ .
- (b) is a straightforward consequence of Lemmas 2.2 and 3.1.
- (c) Let  $V$  be a semisimple Riesz MV-algebra and  $f : A \rightarrow V$  an MV-embedding. Since  $V$  is also divisible, by Remark 2.4, there is a unique MV-embedding  $f_d : \mathbf{A}_d \rightarrow V$  such that

$$f_d \circ \iota_{A,d} \circ \varphi_A = f.$$

If  $\iota_V : V \rightarrow \hat{V}$  is the inclusion of  $V$  in its Dedekind-MacNeille completion, then there exists a unique MV-embedding  $\hat{f}_d : \hat{\mathbf{A}}_d \rightarrow \hat{V}$  such that

$$\hat{f}_d \circ \hat{\iota}_{A,d} = \iota_V \circ f_d.$$



It follows that

$$\hat{f}_d \circ \iota_A = \hat{f}_d \circ \hat{\iota}_{A,d} \circ \iota_{A,d} \circ \varphi_A = \iota_V \circ f_d \circ \iota_{A,d} \circ \varphi_A = \iota_V \circ f,$$

and we get

$$\begin{aligned} \hat{f}_d(\mathbf{R}(A)) &= \hat{f}_d(\langle \iota_A(A) \rangle_{RMV}) = \langle \hat{f}_d(\iota_A(A)) \rangle_{RMV} = \\ &= \langle \iota_V(f(A)) \rangle_{RMV} = \langle f(A) \rangle_{RMV} \subseteq V. \end{aligned}$$

Therefore we define  $f_R : \mathbf{R}(A) \rightarrow V$  as the co-restriction to  $V$  of the restriction  $\hat{f}_d|_{\mathbf{R}(A)}$ . If  $g : \mathbf{R}(A) \rightarrow V$  is another MV-embedding such that  $g \circ \iota_A = f$ , then  $g$  and  $f$  coincide on the generators of  $\mathbf{R}(A)$ , so they coincide on  $\mathbf{R}(A)$ . □

Following [7], we define the Riesz hull of an MV-algebra.

**Definition 4.1.** We say that a Riesz MV-algebra  $U$  is a *Riesz hull* of  $A$  if there exists an essential embedding  $\eta : A \rightarrow U$  such that  $U = \langle \eta(A) \rangle_{RMV}$ .

In consequence, Theorem 4.1 asserts that any semisimple MV-algebra has a Riesz hull which is unique, up to isomorphism.

**Corollary 4.1.** *If  $A$  is a semisimple MV-algebra, then  $\mathbf{R}(A) \simeq \mathbf{R}(\mathbf{A}_d)$ .*

*Proof.* It is a straightforward consequence of the construction. □

**Corollary 4.2.** *If  $A$  is a semisimple MV-algebra and  $V$  is a semisimple Riesz MV-algebra such that  $A \subseteq V$  and  $V = \langle A \rangle_{RMV}$ , then  $V \simeq \mathbf{R}(A)$ .*

*Proof.* By Theorem 4.1 (c), there exists an MV-embedding  $e : \mathbf{R}(A) \rightarrow V$  such that  $e(\iota_A(a)) = a$  for any  $a \in A$ . Hence

$$e(\mathbf{R}(A)) = e(\langle \iota_A(A) \rangle_{RMV}) = \langle e(\iota_A(A)) \rangle_{RMV} = \langle A \rangle_{RMV} = V,$$

so  $e$  is an isomorphism. □

**Corollary 4.3.** *Let  $A$  be a semisimple MV-algebra and  $V$  be a semisimple Riesz MV-algebra such that  $A \subseteq V$  and  $\langle A \rangle_{RMV} = V$ . Then the embedding  $A \hookrightarrow V$  is essential. If, in addition,  $A$  is divisible, then the embedding  $A \hookrightarrow V$  is order dense.*

*Proof.* The first part follows by Corollary 4.2 and Theorem 4.1 (b). If  $A$  is divisible, then  $A \simeq \mathbf{A}_d$ . In this case, the conclusion follows by the fact that  $\mathbf{A}_d \subseteq \mathbf{R}(A) \subseteq \hat{\mathbf{A}}_d$  and Lemma 3.1. □

**Corollary 4.4.** *If  $V$  is a semisimple Riesz MV-algebra, then  $V \simeq \mathbf{R}(V)$ . In this case,  $\iota_V$  is an isomorphism.*

*Proof.* It follows from Corollary 4.2.  $\square$

**Corollary 4.5.** *Assume  $V_1$  and  $V_2$  are semisimple Riesz MV-algebras with the same MV-algebra reduct. Then  $V_1 \simeq V_2$ .*

*Proof.* If  $A$  is the MV-algebra reduct of  $V_1$  and  $V_2$  then, by Corollary 4.2, we get  $V_1 \simeq \mathbf{R}(A) \simeq V_2$ .  $\square$

The above result asserts that, given an MV-algebra  $A$ , there is at most one structure, up to isomorphism, of Riesz MV-algebra with the MV-algebra reduct  $A$ .

In the sequel we prove that the Riesz MV-algebra hull preserves freeness. For a nonempty set  $X$ , we shall denote by  $Free_{MV}(X)$  the free MV-algebra over  $X$  and by  $Free_{RMV}(X)$  the free Riesz MV-algebra over  $X$ . The free algebras exist in the classes of MV-algebras and Riesz MV-algebras since both classes are varieties.

**Proposition 4.1.** *For any nonempty set  $X$ ,  $\mathbf{R}(Free_{MV}(X)) \simeq Free_{RMV}(X)$ . Therefore, the free MV-algebra generated by  $X$  is essentially embedded in the free Riesz MV-algebra generated by  $X$ . Moreover, the embedding can be chosen to be an inclusion.*

*Proof.* If  $T = [0, 1]^{[0, 1]^X}$  then  $T$  is a Riesz MV-algebra with the operations defined component-wise. For any  $x \in X$  we denote by  $\pi_x \in T$  the corresponding projection function and we set  $\tilde{X} = \{\pi_x \mid x \in X\}$ . Since the variety of MV-algebras is generated by  $[0, 1]_{MV}$  and the variety of Riesz MV-algebras is generated by  $[0, 1]_{RMV}$ , by general properties in universal algebra,  $Free_{MV}(X)$  is the MV-algebra generated by  $\tilde{X}$  in  $T$  and  $Free_{RMV}(X)$  is the Riesz MV-algebra generated by  $\tilde{X}$  in  $T$ . We have that  $Free_{MV}(X) = \langle \tilde{X} \rangle_{MV}$  and

$$Free_{RMV}(X) = \langle \tilde{X} \rangle_{RMV} = \langle Free_{MV}(X) \rangle_{RMV}.$$

The conclusion follows from Corollary 4.2.  $\square$

## 5 Categorical setting: the functor $\mathbf{R}$

The main step for obtaining a functorial setting is to prove a general extension result for morphisms, as which we do in Proposition 5.2. The results of this section follow closely the ideas from [3].

**Remark 5.1.** Let  $A$  be a semisimple MV-algebra and  $X \subset A$  such that  $\langle X \rangle_{MV} = A$ . Using Proposition 4.1, the free MV-algebra generated by  $X$  is essentially included in the free Riesz MV-algebra generated by  $X$  and we denote this inclusion by  $\iota_X : Free_{MV}(X) \rightarrow Free_{RMV}(X)$ . Let  $\alpha : Free_{MV}(X) \rightarrow A$  be the unique

MV-homomorphism such that  $\alpha(x) = x$  for any  $x \in X$  and  $\bar{\alpha} : Free_{RMV}(X) \rightarrow R(A)$  be the unique MV-homomorphism such that  $\bar{\alpha}(x) = \iota_A(x)$  for any  $x \in X$ .

$$\begin{array}{ccc} Free_{MV}(X) & \xrightarrow{\iota_X} & Free_{RMV}(X) \\ \alpha \downarrow & & \downarrow \bar{\alpha} \\ A & \xrightarrow{\iota_A} & \mathbf{R}(A) \end{array}$$

**Proposition 5.1.** *Under the above hypothesis, the following properties hold:*

- (a)  $\bar{\alpha} \circ \iota_X = \iota_A \circ \alpha$ ,
- (b)  $\alpha$  and  $\bar{\alpha}$  are surjective,
- (c)  $\ker \bar{\alpha} = \bigcap \{J \mid J \in \mathcal{J}\}$ , where

$$\mathcal{J} = \{J \subseteq Free_{RMV}(X) \mid J \text{ ideal, } \iota_X(\ker \alpha) \subseteq J, \text{ and } Free_{RMV}(X)/J \text{ is semisimple}\}.$$

*Proof.*

(a)  $(\bar{\alpha} \circ \iota_X)(x) = \iota_A(x) = (\iota_A \circ \alpha)(x)$  for any  $x \in X$ , so the morphisms coincide on generators.

(b)  $\alpha(Free_{MV}(X)) = \alpha(\langle X \rangle_{MV}) = \langle \alpha(X) \rangle_{MV} = A$  and

$$\begin{aligned} \bar{\alpha}(Free_{RMV}(X)) &= \bar{\alpha}(\langle Free_{MV}(X) \rangle_{RMV}) = \\ &\langle \iota_A(\alpha(Free_{MV}(X))) \rangle_{RMV} = \langle \iota_A(A) \rangle_{RMV} = \mathbf{R}(A). \end{aligned}$$

(c) If  $z \in \ker \alpha$  then  $\iota_A(\alpha(z)) = 0$ , so  $\bar{\alpha}(\iota_X(z)) = 0$ . It follows that  $\iota_X(\ker \alpha) \subseteq \ker \bar{\alpha}$ . In fact, we have  $\iota_X(\ker \alpha) = \ker \bar{\alpha} \cap \iota_X(Free_{MV}(X))$ . We set  $\bar{J} = \bigcap \{J \mid J \in \mathcal{J}\}$  and  $F = Free_{RMV}(X)/\bar{J}$ . By a general result of universal algebra [6, Proposition 7.1],  $F$  is isomorphic with a subdirect product of the family  $\{Free_{RMV}(X)/J \mid J \in \mathcal{J}\}$ , so  $F$  is a subalgebra of a direct product of semisimple MV-algebras. Therefore,  $F$  is a semisimple MV-algebra. If we set  $M = \{y/\bar{J} \mid y \in \iota_X(Free_{MV}(X))\}$  then  $\langle M \rangle_{RMV} = F$ , so  $\mathbf{R}(M) = F$  by Corollary 4.3 and the inclusion  $M \subseteq F$  is essential.

It is clear that  $\iota_X(\ker \alpha) \subseteq \bar{J} \subseteq \ker \bar{\alpha}$ . In order to prove that  $\bar{J} = \ker \bar{\alpha}$ , we assume that there exists an element  $z \in \ker \bar{\alpha} \setminus \bar{J}$ . Hence  $z/\bar{J} \neq 0$  in  $F$ . Since the inclusion  $M \subseteq F$  is essential it follows that there exists an element  $y \in \iota_X(Free_{MV}(X))$  such that  $0 < y/\bar{J} \leq nz/\bar{J}$ . Note that  $y/\bar{J} \neq 0$  in  $F$ .

We denote  $w = y \odot (nz)^*$ , so  $w \in \overline{J}$  and  $y \leq (nz) \vee (y) = (nz) \oplus w$ . Note that  $w \in \overline{J} \subseteq \ker \overline{\alpha}$  and  $z \in \ker \overline{\alpha}$ , so we get  $y \in \ker \overline{\alpha}$ . But  $y \in \iota_X(\text{Free}_{MV}(X))$ , so  $y \in \iota_X(\text{Free}_{MV}(X)) \cap \ker \overline{\alpha} = \iota_X(\ker \alpha)$ . Since  $\iota_X(\ker \alpha) \subseteq \overline{J}$ , it follows that  $y/\overline{J} = 0$  in  $F$ , which is a contradiction.

□

**Proposition 5.2.** *Let  $A$  be a semisimple MV-algebra. For any semisimple Riesz MV-algebra  $V$  and for any MV-homomorphism  $f : A \rightarrow V$  there exists a unique MV-homomorphism  $f_R : \mathbf{R}(A) \rightarrow V$  such that  $f_R(\iota_A(a)) = f(a)$ , for any  $a \in A$ .*

$$\begin{array}{ccc} A & \xrightarrow{\iota_A} & \mathbf{R}(A) \\ & \searrow f & \swarrow \text{dotted } f_R \\ & V & \end{array}$$

*Proof.* Assume  $V$  is a semisimple Riesz MV-algebra and  $f : A \rightarrow V$  is an MV-homomorphism. We consider  $X \subseteq A$  such that  $\langle X \rangle_{MV} = A$  and we define  $\alpha : \text{Free}_{MV}(X) \rightarrow A$  and  $\overline{\alpha} : \text{Free}_{RMV}(X) \rightarrow \mathbf{R}(A)$  as in Remark 5.1. Let  $\overline{f} : \text{Free}_{RMV}(X) \rightarrow V$  be the unique MV-homomorphism such that  $\overline{f}(x) = f(x)$  for any  $x \in X$ . By Proposition 5.1 (b), we infer that  $\text{Free}_{RMV}(X)/\ker \overline{\alpha} \simeq \mathbf{R}(A)$  and we can safely identify them.

$$\begin{array}{ccccc} & & \text{Free}_{MV}(X) & & \\ & \swarrow \alpha & & \searrow \iota_X & \\ A & \xrightarrow{\iota_A} & \mathbf{R}(A) & \xleftarrow{\overline{\alpha}} & \text{Free}_{RMV}(X) \\ & \searrow f & \downarrow \text{dotted } f_R & \swarrow \overline{f} & \\ & & V & & \end{array}$$

We note that  $\overline{f}(\text{Free}_{RMV}(X))$  is a Riesz MV-subalgebra of  $V$ , so it is semisimple. Therefore, by Proposition 5.1 (c) it follows that  $\ker \overline{\alpha} \subseteq \ker \overline{f}$ , so there exists a unique MV-homomorphism  $f_R : \mathbf{R}(A) \rightarrow V$  such that  $f_R \circ \overline{\alpha} = \overline{f}$ . It follows that

$$f_R \circ \iota_A \circ \alpha = f_R \circ \overline{\alpha} \circ \iota_X = \overline{f} \circ \iota_X = f \circ \alpha.$$

Since  $\alpha$  is surjective, we get  $f_R \circ \iota_A = f$ .

In order to prove the uniqueness, assume that  $g : \mathbf{R}(A) \rightarrow V$  is an MV-homomorphism such that  $g \circ \iota_A = f$ . It follows that  $g \circ \overline{\alpha} \circ \iota_X = g \circ \iota_A \circ \alpha = f \circ \alpha = \overline{f} \circ \iota_X$  and we get  $g \circ \overline{\alpha} = \overline{f}$ , since they coincide on the generators of  $Free_{RMV}(X)$ . We proved that  $g$  satisfies the property that uniquely defines  $f_R$ , so  $g = f_R$ .  $\square$

**Lemma 5.1.** *Let  $A$  and  $B$  be semisimple MV-algebras. For any homomorphism  $h : A \rightarrow B$ , there is a unique homomorphism  $\mathbf{R}(h) : \mathbf{R}(A) \rightarrow \mathbf{R}(B)$  such that*

$$\mathbf{R}(h) \circ \iota_A = \iota_B \circ h.$$

*In addition, if  $h$  is an embedding, then  $\mathbf{R}(h)$  is also an embedding.*

$$\begin{array}{ccc} A & \xrightarrow{\iota_A} & \mathbf{R}(A) \\ h \downarrow & & \downarrow \mathbf{R}(h) \\ B & \xrightarrow{\iota_B} & \mathbf{R}(B) \end{array}$$

*Proof.* We apply Proposition 5.2 for  $V = \mathbf{R}(B)$  and for  $f = \iota_B \circ h$ . Therefore  $\mathbf{R}(h) = f_R$ .  $\square$

We consider the forgetful functor between the category  $\mathcal{RMV}_s$  of semisimple RMV-algebras and the category  $\mathcal{MV}_s$  of semisimple MV-algebras:

$$\mathbf{U} : \mathcal{RMV}_s \rightarrow \mathcal{MV}_s,$$

which forgets the scalar multiplication.

We also define the functor

$$\mathbf{R} : \mathcal{MV}_s \rightarrow \mathcal{RMV}_s$$

as follows:

- for any semisimple MV-algebra  $A$ ,  $\mathbf{R}(A)$  is the Riesz hull of  $A$ ,
- for any MV-homomorphism  $h : A \rightarrow B$ ,  $\mathbf{R}(h)$  is the unique homomorphism such that  $\mathbf{U}(\mathbf{R}(h)) \circ \iota_A = \iota_B \circ h$ .

**Theorem 5.1.** *Under the above settings,  $(\mathbf{R}, \mathbf{U})$  is an adjoint pair.*

*Proof.* One can easily see that  $\mathbf{R}$  is a functor. If  $A$  is a semisimple MV-algebra we define  $\eta_A : A \rightarrow \mathbf{U}(\mathbf{R}(A))$ ,  $\eta_A(a) = \iota_A(a)$  for any  $a \in A$ . If  $V$  is a semisimple Riesz MV-algebra, let  $\varepsilon_V = \iota_V^{-1} : \mathbf{R}(\mathbf{U}(V)) \rightarrow V$ . By Corollary 4.4,  $\varepsilon_V$  is an isomorphism.

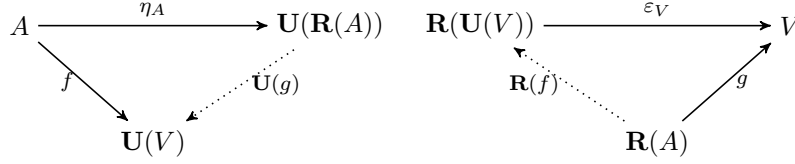
In order to prove that  $\mathbf{R}$  is a left adjoint to  $\mathbf{U}$ , we have to prove the following properties, for any MV-algebra  $A$  and Riesz MV-algebra  $V$ :

(1) for any  $f \in \mathcal{MV}_s(A, \mathbf{U}(V))$ , there exists  $g \in \mathcal{RMV}_s(\mathbf{R}(A), V)$  such that

$$\mathbf{U}(g) \circ \eta_A = f,$$

(2) for any  $g \in \mathcal{RMV}_s(\mathbf{R}(A), V)$  there exists  $f \in \mathcal{MV}_s(A, \mathbf{U}(V))$  such that

$$\varepsilon_V \circ \mathbf{R}(f) = g.$$



The property (1) follows by Proposition 5.2 with  $g = f_R$  whenever  $f \in \mathcal{MV}_s(A, \mathbf{U}(V))$ . In order to prove (2), assume that  $g \in \mathcal{RMV}_s(\mathbf{R}(A), V)$  and set  $f = \mathbf{U}(g) \circ \iota_A$ . Hence  $\mathbf{R}(f)$  is the unique homomorphism such that

$$\mathbf{U}(\mathbf{R}(f)) \circ \iota_A = \iota_{\mathbf{U}(V)} \circ f.$$

Therefore we have  $\mathbf{U}(\mathbf{R}(f)) \circ \iota_A = \iota_{\mathbf{U}(V)} \circ \mathbf{U}(g) \circ \iota_A$ . Since  $\iota_A$  is an embedding, we get that  $\mathbf{U}(\mathbf{R}(f)) = \iota_{\mathbf{U}(V)} \circ \mathbf{U}(g)$ .

We note that  $\iota_{\mathbf{U}(V)} = \mathbf{U}(\iota_V) = \mathbf{U}(\varepsilon_V^{-1})$ . It follows that  $\mathbf{U}(\varepsilon \circ \mathbf{R}(f)) = \mathbf{U}(g)$ , so  $\varepsilon \circ \mathbf{R}(f) = g$ .  $\square$

In the following we prove that the hull functor  $\mathbf{R}$  and the functor  $\Gamma$  commute.

**Remark 5.2.** Let  $A$  be a semisimple MV-algebra and  $(G, u)$  an  $\ell u$ -group such that  $A = \Gamma(G, u)$ . If  $X \subseteq A$  and  $\langle X \cup \{u\} \rangle_\ell$  is the  $\ell$ -group generated by  $X \cup \{u\}$  in  $G$ , then we note that  $(\langle X \cup \{u\} \rangle_\ell, u)$  is an  $\ell u$ -subgroup of  $(G, u)$ . As a consequence, we infer that  $\Gamma(\langle X \cup \{u\} \rangle_\ell, u)$  is an MV-subalgebra of  $A$ . It is now straightforward that  $\langle X \rangle_{MV} = \Gamma(\langle X \cup \{u\} \rangle_\ell, u)$ . Assume now that  $V$  is a Riesz MV-algebra,  $(H, u)$  is a unital Riesz space such that  $\Gamma_R(H, u) = V$  and  $X \subseteq V$ . It is straightforward that  $\langle X \rangle_{RMV} = \Gamma_R(\langle X \cup \{u\} \rangle_{v\ell}, u)$ , where  $\langle X \cup \{u\} \rangle_{v\ell}$  is the Riesz space generated by  $X \cup \{u\}$  in  $H$ .

**Proposition 5.3.** *If  $A$  is a semisimple MV-algebra and  $(G, u)$  an  $\ell u$ -group such that  $A = \Gamma(G, u)$ , then  $\mathbf{R}(A) = \Gamma_R(\mathbf{R}(G), u)$ .*

*Proof.* We recall that  $\mathbf{A}_d = \Gamma(\mathbf{G}_d, u)$ , so  $\mathbf{A}_d \subseteq \mathbf{G}_d \subseteq \hat{\mathbf{G}}_d$ . Moreover,  $\hat{\mathbf{G}}_d$  is a Riesz space and  $\mathbf{R}(G) = \langle \mathbf{G}_d \rangle_{v\ell}$ , i.e.  $\mathbf{R}(G)$  is the Riesz space generated by  $\mathbf{G}_d$  in  $\hat{\mathbf{G}}_d$ . Following Remark 5.2 we have  $\mathbf{R}(A) = \langle \mathbf{A}_d \rangle_{RMV} = \Gamma_R(\langle \mathbf{A}_d \cup \{u\} \rangle_{v\ell}, u)$ . Since  $u \in \mathbf{A}_d$  and  $\langle \mathbf{A}_d \rangle_{v\ell} = \langle \mathbf{G}_d \rangle_{v\ell}$ , we get

$$\mathbf{R}(A) = \Gamma_R(\langle \mathbf{G}_d \rangle_{v\ell}, u) = \Gamma_R(\mathbf{R}(G), u).$$

□

We denote by  $\mathcal{AG}_{ua}$  the category of archimedean  $\ell u$ -groups and by  $\mathcal{RS}_{ua}$  the category of archimedean Riesz spaces with strong unit. By [3], the correspondence  $G \mapsto \mathbf{R}(G)$ , which associates to an  $\ell$ -group its  $v$ -hull, is functorial. If  $G$  has a strong unit  $u$ , following Conrad's construction, one can easily see that  $u$  is also a strong unit of  $\mathbf{R}(G)$ . Hence we get a functor  $\mathbf{R} : \mathcal{AG}_{ua} \rightarrow \mathcal{RS}_{ua}$ .

**Theorem 5.2.** *The following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{AG}_{ua} & \xrightarrow{\Gamma} & \mathcal{MV}_s \\ \mathbf{R} \downarrow & & \downarrow \mathbf{R} \\ \mathcal{RS}_{ua} & \xrightarrow{\Gamma_R} & \mathcal{RMV}_s \end{array}$$

*Proof.* It is a straightforward consequence of Proposition 5.3. □

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